

PHASE SPACE LOCALIZATION FOR ANTI-DE SITTER QUANTUM MECHANICS AND ITS ZERO CURVATURE LIMIT

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Abstract

Using techniques of geometric quantization and $SO_0(3,2)$ -coherent states, a notion of optimal localization on phase space is defined for the quantum theory of a massive and spinning particle in anti-de Sitter spacetime. We show that this notion disappears in the zero curvature limit, providing one with a concrete example of the regularizing character of the constant (nonzero) curvature of the anti-de Sitter spacetime. As a byproduct a geometric characterization of masslessness is obtained.

The present contribution is based on a joint work with Stephan De Bièvre (see references quoted below).

1 Introduction

It is a well known fact that the Poincaré group, $\mathcal{P}_+^{\uparrow}(3,1)$, the kinematical group of Minkowski spacetime, can be obtained by means of a contraction from the anti-de Sitter (AdS) group, $SO_0(3,2)$, the kinematical group of anti-de Sitter spacetime. The contraction parameter is the constant positive curvature κ of the anti-de Sitter spacetime. This contraction procedure is thus nothing but a zero curvature limit. According to this fact, one would like to approximate $\mathcal{P}_+^{\uparrow}(3,1)$ -invariant theories by $SO_0(3,2)$ -invariant ones, hoping that such approximations give rise to regularized relativistic theories [1] [2]. Indeed, the nonzero curvature equips the AdS theories with a lengthlike parameter, which is actually the source of the sought regularizations.

Up to now, this very stimulating idea has not been fully exploited, though it has received a large amount of attention for its potential implications in the context of quantum field theories. The main drawback of the known approaches arises from the emphasis made on the spacetime or the momentum space realizations of those theories. Indeed, it is a known fact that such realizations, in both Poincaré and AdS cases, lack of a natural notion of localization. Moreover the modulus of the wave functions corresponding to the one particle quantum states of a Poincaré, as well as an AdS, free massive theory can not be interpreted, in those realizations, as a probability distribution. The regularizing role of κ is thus not effective for such realizations.

In this short contribution we propose the phase space realization as the regularizing alternative. In fact, for the case of a free massive spinning particle in AdS spacetime, the phase space is a Kähler $SO_0(3,2)$ homogeneous space, whose (geometric) quantization gives rise to a discrete series representation of $SO_0(3,2)$. The latter is known to be a square integrable representation, so its Hilbert space contains a particular family of quantum states: the coherent states. A natural notion of localization is attached to these states. They are *optimally localized* states in phase space. Moreover the modulus of the wave functions of the quantum states in this realization can be actually interpreted as a probability distribution.

Here we exhibit the explicit form of these coherent states and we show how their physical interpretation arises. We also stress the disappearance of this notion of localization in the flat space limit, confirming the effectiveness of the regularizing character of κ . We proceed as follows. In section 2 we describe the classical theory, in order to fix both the notations and the physical interpretations. In section 3, the quantum theory is obtained through the application of geometric quantization, then the explicit form and the zero curvature limit of the optimally localized states is given. Section 4 contains a brief discussion of a geometric characterization of masslessness as it arises from the description of section 2. For more details we refer to the papers [3], [4], [5] and [6].

2 The classical theory

The phase space description of the classical theory of a spin s and mass $m \neq 0$ free particle in AdS spacetime finds its best formulation within the scheme developed by Souriau [7]. The latter construction starts with the determination of an *evolution space*, $(E_\kappa^{m,s}, \omega_E)$, which is a presymplectic manifold (ω_E is a closed but degenerate 2-form), with a projection on the AdS spacetime of constant curvature κ , M_κ . The symmetries of M_κ are helpful guides in doing so. In fact, M_κ is just the one sheeted hyperboloid in (\mathbf{R}^5, η) , with $\text{diag } \eta = \begin{pmatrix} 5 & 0 & 1 & 2 & 3 \\ - & - & + & + & + \end{pmatrix}$,

$$y \cdot y \equiv \eta_{\alpha\beta} y^\alpha y^\beta = -(y^5)^2 - (y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 = -\kappa^{-2}, \quad (2.1)$$

$\alpha, \beta \in \{5, 0, 1, 2, 3\}$. Clearly, $O(3,2)$ is the isometry group of (2.1), its connected component to the identity. $SO_0(3,2)$, is the so-called AdS group.

We choose for $E_\kappa^{m,s}$ the $SO_0(3,2)$ -principal homogeneous space, $E_\kappa^{m,s} \cong SO_0(3,2)$, realized through the following $SO_0(3,2)$ -invariant constraints in \mathbf{R}^{25} (five copies of (\mathbf{R}^5, η)),

$$y \cdot y = -\kappa^{-2}, \quad q \cdot q = -m^2, \quad u \cdot u = 1, \quad v \cdot v = 1 \quad \text{and} \quad t \cdot t = m^2 s^2, \quad (2.2a)$$

$$y \cdot q = 0 = \text{all the other scalar products} \quad (2.2b)$$

$$\epsilon_{\alpha\beta\gamma\rho\sigma} y^\alpha q^\beta u^\gamma v^\rho t^\sigma = \frac{m^2 s}{\kappa} \quad \text{and} \quad y^5 q^0 - y^0 q^5 > 0. \quad (2.2c)$$

The physical interpretation of the coordinates (y, q, u, v, t) is then as follows: in (2.2a) y is the position on the hyperboloid (2.1), q is its conjugate momentum, t is what we call the AdS-Pauli-Lubanski vector. The remaining five-vectors u and v are introduced in order to have a covariant description of $E_\kappa^{m,s}$, i.e. $E_\kappa^{m,s} \cong SO_0(3,2)$. They shall represent the spin part in the quantum theory. The two last constraints (2.2b-c) are needed in order to fix an orientation.

The choice of ω_E is constrained by the requirement that the projection on M_κ of each integral curve of the completely integrable distribution generated by $\ker \omega_E$ in $E_\kappa^{m,s}$, results in a time-like geodesic of M_κ , i.e. the dynamic of the theory is obtained from $\ker \omega_E$. Such an ω_E is provided by,

$$\omega_E = dy \wedge dq + s du \wedge dv. \quad (2.3)$$

This choice is not unique but it fulfils the above dynamic generating requirement. The phase space of the theory, $(\Sigma_\kappa^{m,s}, \omega_E)$, is obtained by symplectic reduction of $(E_\kappa^{m,s}, \omega_E)$. It appears, for $\frac{m}{\kappa} \neq s$, to be the $SO_0(3,2)$ symplectic homogeneous space $SO_0(3,2)/SO(2) \times SO(2)$. For symmetry reasons, i.e. obvious action of $SO_0(3,2)$ on $E_\kappa^{m,s}$, we use $(E_\kappa^{m,s}, \omega_E)$ as the arena for the forthcoming constructions. The special case $\frac{m}{\kappa} = s$ is discussed in section 4.

In order to carry out the zero curvature limit in a meaningful way, we introduce a new set of coordinates on $E_\kappa^{m,s}$. This is the set of four-vectors (x, p, a, b, s) . Interpreted in the same way as the five-vectors (y, q, u, v, t) , they are related to the latter through the following equations,

$$y^5 = Y \cos \kappa x^0, \quad y^0 = Y \sin \kappa x^0 \quad \text{and} \quad \vec{y} = \vec{x}, \quad (2.4a)$$

where $-\pi \leq \kappa x^0 \leq \pi$, $\vec{x} \in \mathbf{R}^3$ and $Y = \sqrt{\kappa^{-2} + (\vec{x})^2}$; and

$$q \cdot dy = g_{\mu\nu} p^\mu dx^\nu, \quad u \cdot dy = g_{\mu\nu} a^\mu dx^\nu, \quad v \cdot dy = g_{\mu\nu} b^\mu dx^\nu \quad \text{and} \quad t \cdot dy = g_{\mu\nu} s^\mu dx^\nu. \quad (2.4b)$$

Here $g_{\mu\nu}$ is the metric of M_κ for the global coordinates (x^0, \vec{x}) and $\mu, \nu \in \{0, 1, 2, 3\}$. The zero curvature limit of $g_{\mu\nu}$ is just the flat Minkowski metric. The constraints (2.2a-c) translated in terms of the new coordinates become,

$$g_{\mu\nu} p^\mu p^\nu = -m^2, \quad g_{\mu\nu} a^\mu a^\nu = 1, \quad g_{\mu\nu} b^\mu b^\nu = 1 \quad \text{and} \quad g_{\mu\nu} s^\mu s^\nu = m^2 s^2, \quad (2.5a)$$

$$g_{\mu\nu} p^\mu s^\nu = 0 = \text{all the other scalar products of the subset } (p, a, b, s), \quad (2.5b)$$

$$\epsilon_{\mu\nu\lambda\delta} p^\mu a^\nu b^\lambda s^\delta = m^2 s \quad \text{and} \quad p^0 > 0. \quad (2.5c)$$

The physical interpretation of the above constraints can now be confirmed by their zero curvature limits.

3 The quantum theory and the optimal localization

The methods of geometric quantization allow one to quantize the classical theory described above [8]. In other words, using those methods one is able to construct the unitary irreducible representation of $SO_0(3,2)$ associated to the coadjoint orbit of $SO_0(3,2)$ for which the phase space $\Sigma_\kappa^{m,s}$ is a covering. Exploiting the principal bundle structure $E_\kappa^{m,s} \cong SO_0(3,2) \rightarrow SO_0(3,2)/SO(2) \times SO(2) \cong \Sigma_\kappa^{m,s}$, the prequantum Hilbert space, \mathcal{H} , is realized as follows,

$$\mathcal{H} = \left\{ \psi : E_\kappa^{m,s} \rightarrow \mathbf{C} \mid \int_{E_\kappa^{m,s}} |\psi|^2 d\mu_\kappa^{m,s} < \infty, \quad Y_{50}\psi = i\frac{m}{\kappa}\psi \quad \text{and} \quad Y_{12}\psi = is\psi \right\}. \quad (3.1)$$

Here $d\mu_\kappa^{m,s}$ is the invariant measure on $E_\kappa^{m,s}$ and Y_{50} and Y_{12} are the left invariant vector fields generating $\ker \omega_E$. Since $E_\kappa^{m,s} \cong SO_0(3,2)$, there exists a natural action of $SO_0(3,2)$ in $L^2(E_\kappa^{m,s}, d\mu_\kappa^{m,s})$. This yields the left regular representation of $SO_0(3,2)$. The latter restricts

to a unitary (reducible) representation in \mathcal{H} , i.e. the representation of $SO_0(3, 2)$ induced by the character $e^{i(\frac{m}{\kappa}\tau + s\tau')}$ of $SO(2) \times SO(2)$. Indeed, this holds provided $\frac{m}{\kappa}$ and s are both integers.

There actually exists a positive invariant Kählerian polarization of $\Sigma_\kappa^{m,s}$ allowing one to select in \mathcal{H} an invariant subspace $\mathcal{H}_\kappa^{m,s}$. The restriction of the previous unitary representation to the latter gives rise to a unitary irreducible representation of $SO_0(3, 2)$. Concretely,

$$\mathcal{H}_\kappa^{m,s} = \left\{ \psi \in \mathcal{H} \mid \bar{Z}_i \psi = 0, \quad i \in \{1, 2, 3\} \quad \text{et} \quad \bar{\Xi} \psi = 0 \right\}; \quad (3.2)$$

where $Z_i = Y_{0i} + iY_{i5}$, $i \in \{1, 2, 3\}$ and $\Xi = Y_{23} + iY_{31}$. The $Y_{\alpha\beta}$'s are the left invariant vector fields. The way one obtains the unitary irreducible representation carried by $\mathcal{H}_\kappa^{m,s}$ is known in the mathematic litterature as the holomorphic induction, it yields the discrete series representation of $SO_0(3, 2)$ with highest weight $(\frac{m}{\kappa}, s)$. (A necessary condition for the unitarity is $\frac{m}{\kappa} > s$.)

The quantum states of the theory are represented by well defined wave functions belonging to $\mathcal{H}_\kappa^{m,s}$. The physical interpretation of their modulus as probability distributions on $\Sigma_\kappa^{m,s}$ is also well defined. The particular states belonging to the orbit, $\mathcal{O}_{\varphi_0} \subset \mathcal{H}_\kappa^{m,s}$, of the unitary representation of $SO_0(3, 2)$ passing through the highest weight state φ_0 possess many interesting properties [9]. These states, which are nothing but the generalized coherent states of $SO_0(3, 2)$, are in a natural way optimally localized in phase space. In fact, by construction they are labeled by points $w \in E_\kappa^{m,s}$, specifying them through the equations,

$$\langle \varphi_w \mid \hat{L}_{\alpha\beta} \mid \varphi_w \rangle = L_{\alpha\beta}(w), \quad \forall \alpha, \beta \in \{5, 0, 1, 2, 3\}; \quad (3.3)$$

here the $L_{\alpha\beta}$'s are the classical observables and the $\hat{L}_{\alpha\beta}$'s are their quantum counterparts. The determination through (3.3) of the ten $L_{\alpha\beta}(w)$ specifies in fact uniquely the leave of the distribution $\ker \omega_E$ passing through w . Thus by symplectic reduction a unique point $\bar{w} \in \Sigma_\kappa^{m,s}$ is specified by (3.3). The state φ_w is then said to be localized in $\bar{w} \in \Sigma_\kappa^{m,s}$. Moreover, since the coherent states minimize the uncertainty relations associated to the commutation relations of the $\hat{L}_{\alpha\beta}$'s, this notion of localization is then optimal.

The optimally localized states are given by the following formula,

$$\varphi_{z', \xi'}(z, \xi) = (-2)^{\frac{m}{\kappa}} (2)^{-s} (\bar{z}' \cdot z)^{-\frac{m}{\kappa} - s} \left[(\bar{z}' \cdot z)(\bar{\xi}' \cdot \xi) - (\bar{z}' \cdot \xi)(\bar{\xi}' \cdot z) \right]^s. \quad (3.4)$$

Here $(z, \xi) \equiv w$ are the complex coordinates of $E_\kappa^{m,s}$ associated to the Kählerian polarization, they are related to the coordinates given in (2.2) through the transformations $z = \kappa y - im^{-1} q$ and $\xi = u - iv$.

The zero curvature limit of these states is as follows,

$$\lim_{\kappa \rightarrow 0} \left(\frac{m}{4\pi\kappa} \right)^{\frac{3}{2}} \varphi_{z', \xi'}(z, \xi) = m^2 p^0 \delta(\bar{p} - \bar{p}') e^{-ip_\mu(x'^\mu - x^\mu)} \left(\frac{\bar{\zeta}' \cdot \zeta}{2} \right)^s, \quad (3.5)$$

where $\zeta_\mu = a_\mu - ib_\mu$, $\mu \in \{0, 1, 2, 3\}$. Clearly, these states are no longer optimally localized. They are completely delocalized in position (x), perfectly localized in momentum (p) and still optimally localized in spin (ζ). This zero curvature behaviour supports the regularization argument stressed in the introduction. In fact one can consider the AdS states in (3.4) as regularizations of the (generalized) Poincaré states in (3.5).

4 Remarks on masslessness

When evaluating $\ker \omega_E$ in section 2 two possibilities actually arises. Either $\frac{m}{\kappa} = s$ or $\frac{m}{\kappa} \neq s$. We have dealt here and in [3], [4] and [5] only with the second case, which corresponds to a massive elementary system. In fact, when $\frac{m}{\kappa} \neq s$ $\dim \ker \omega_E = 2$ and then $\dim \Sigma_{\kappa}^{m,s} = 8$, since $\Sigma_{\kappa}^{m,s} \equiv E_{\kappa}^{m,s} / \ker \omega_E$. This eight dimensional phase space becomes in the zero curvature limit an eight dimensional Poincaré ($\mathcal{P}_+^{\uparrow}(3,1)$) phase space [4] [5]. It is well known [7] that only the massive ($m \neq 0$) and spinning ($s \neq 0$) free particles on Minkowski spacetime have their dynamics described by an eight dimensional $\mathcal{P}_+^{\uparrow}(3,1)$ -invariant phase space. This confirms the fact that the AdS systems for which $\frac{m}{\kappa} \neq s$ describe massive AdS elementary systems.

The situation is different when $\frac{m}{\kappa} = s$. Actually, in this case $\dim \ker \omega_E = 4$ and then $\dim \Sigma_{\kappa}^{m,s} = 6$. The limiting theory is clearly no longer the ($m \neq 0, s \neq 0$) $\mathcal{P}_+^{\uparrow}(3,1)$ -invariant one (the limiting phase space is also six dimensional). The known six dimensional $\mathcal{P}_+^{\uparrow}(3,1)$ -invariant phase spaces are those associated to the ($m = 0, s \neq 0$) and ($m \neq 0, s = 0$) free particles on Minkowski spacetime. Since the contraction $SO_0(3,2) \rightarrow \mathcal{P}_+^{\uparrow}(3,1)$ preserves the $SO_0(3,1)$ Lorentz subgroup, the spin part is not altered in the zero curvature limit. Hence, the obvious candidate for the limiting theory is the ($m = 0, s \neq 0$) $\mathcal{P}_+^{\uparrow}(3,1)$ -invariant system. Thus, the AdS systems such that $\frac{m}{\kappa} = s$ describe massless AdS elementary systems. A deeper analysis of this phenomenon is addressed elsewhere [6]. There one can find a more rigorous treatment.

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